STATEMENT REPORT

under the procedure for acquisition of the scientific degree "Doctor of Science"

by candidate: **Borislav Radkov Draganov**
of the DSci Thesis entitled: **"Simultaneous Approximation by the Bernstein Operator"**
In the Scientific field: **4. Natural Sciences, Mathematics and Informatics**
Professional field: **4.5. Mathematics**
Doctoral program: **"Mathematical Analysis"**, Department: **"Mathematical Analysis"**
Faculty of Mathematics and Informatics (FMI), **Sofia University "St. Kl. Ohridski" (SU)**

The statement report has been prepared by: Assoc. Prof. PhD, Rumen Uluchev, FMI, SU, as a member of the scientific jury for the defense of this DSci Thesis according to Order № РД 38-627/28.11.2023 of the Rector of the Sofia University.

1. **General characteristics of the dissertation thesis and the presented materials**

   Borislav Draganov have submitted a DSci Thesis entitled *Simultaneous Approximation by the Bernstein Operator*, in English. The main object of his research are estimates in weighted norms for the simultaneous approximation of sufficiently smooth functions and their derivatives by the Bernstein operator or its modifications. The thesis (178 pages) consists of an introduction, 6 chapters and a bibliography containing 100 cited research publications. An abstract, 9 scientific publications, a citations list of these publications, etc. are also submitted to the procedure.

2. **Short CV and personal impressions of the candidate**

   The Candidate, Borislav Draganov has held the following academic positions in the Department of Mathematical Analysis, FMI, SU: he was elected Assistant Professor in 2002, since 2004 he has been Senior Assistant Professor, and since 2011 - Associate Professor. He was awarded PhD in 2003. In addition, since 2012 he is Associate Professor at IMI-BAS.
   I know well about the scientific activities and results of Borislav Draganov since we participated in many conferences and workshops in Bulgaria and abroad, where he has presented his results, from our joint work in scientific projects funded by the NSF of the Ministry of Education and Science, the NSF of SU, the EU through the National Recovery and Resilience Plan of the Republic of Bulgaria, from our joint editorial work on the conference proceedings "Constructive Theory of Functions", etc. I have excellent impressions of Borislav Draganov as a colleague, researcher and teacher

3. **Content analysis of the scientific and applied achievements of the candidate, contained in the presented PhD thesis and the publications to it, included in the procedure**

   The property of Bernstein polynomials $B_n f(x)$ that for natural s and sufficiently smooth functions f we have

   $$\lim_{n \to \infty} (B_n f)^{(s)}(x) = f^{(s)}(x)$$

uniformly on [0,1], has been known since the 1930s. This interesting property is said to be simultaneous approximation and is in the basis of the problems under consideration in this dissertation. I will emphasize some of the most important results in the thesis.

Chapter 1 is preparatory and contains basic definitions of the K-functionals, the moduli of smoothness and their properties to be used later. In Chapter 2 some inequalities for norms (including weights) of differential operators of intermediate order are proved. The resulting inequalities are further applied in the proofs of a number of results. In Chapter 3 the problem of simultaneous approximation of functions by the polynomials $B_n f(x)$ is considered and appropriate K-functionals are used to characterize the approximation. A direct inequality is proved in Theorem 3.3. Namely, an estimate from above is obtained for the weighted error $\| w(B_n f - f)^{(s)} \|$ (the essential supremum at Jacobi weight $w(x)$) by using K-functionals. In Theorem 3.5, Jackson-type estimates are derived for various s and weights w. The strong converse inequalities proved in Theorem 3.8 show that the estimates of Theorem 3.3 are sharp in rate. Moreover, under some restrictions on s and the weight w, in Theorem 3.30 a complete characterization of the simultaneous approximation error using K-functionals is obtained.

Chapter 4 extends results from the previous one, for iterated Boolean sums of the Bernstein operator $\mathbf{B}_{r,n} = I - (I - B_n)^r$, where r is a natural number. Thus, in Theorem 4.3 a direct inequality for the weighted error $\| w(\mathbf{B}_{r,n} f - f)^{(s)} \|$ and appropriate K-functional is proved. Theorem 4.7 and Theorem 4.8 give Jackson-type estimates, while a converse inequality is obtained in Theorem 4.10.

In Chapter 5, Borislav Draganov studies a simultaneous approximation with two modifications of Bernstein polynomials with integer coefficients of the basis polynomials. The operator $\tilde{B}_n(f)(x) = \sum_{k=0}^{n} \lfloor f\left(\frac{k}{n}\right) \binom{n}{k} \rfloor x^k (1-x)^{n-k}$ was introduced by Kantorovich and the operator $\hat{B}_n(f)(x) = \sum_{k=0}^{n} \langle f\left(\frac{k}{n}\right) \binom{n}{k} \rangle x^k (1-x)^{n-k}$, $\langle a \rangle$ denotes the closest integer to the real number a, was suggested by Draganov. For the error estimates of the simultaneous approximation by using above defined versions of the Bernstein operator, direct inequalities with moduli of smoothness (Theorem 5.1 and Theorem 5.4) and weak converse inequalities (Theorem 5.6) are established.

Under the notations $D_n f(x) = n\big(B_n f(x) - f(x)\big)$ and $\boldsymbol{D}f(x) = \frac{\varphi^2(x)}{2} f''(x)$, where $\varphi^2(x) = x(1-x)$, the Voronovskaya theorem claims that for functions f of the space $C^2[0,1]$ the linear operator $D_n f(x)$ converges to the differential operator $\boldsymbol{D}f(x)$ uniformly on [0,1]. In Chapter 6 the Candidate investigates the error of approximation $D_n f(x) \approx \boldsymbol{D}f(x)$, i.e. the rate of convergence in Voronovskaya theorem. There are proved in Theorem 6.1 direct and weak converse inequalities with estimates in appropriate K-functionals. A characterization of the error $\| D_n f - \boldsymbol{D}f \|$ follows immediately in Corollary 6.3 from these estimates.

4. Approbation of the results

All the submitted 9 papers (one of which is co-authored with a declaration of equal participation of both authors) were published between 2013 and 2022 in peer reviewed editions as follows: *Journal of Approximation Theory* (3 articles, total IF: 2,837), *Results in Mathematics* (2, total IF: 1,737), *Studia Universitatis Babeş-Bolyai Mathematica* (1, IF: 0,4), *Annual of Sofia University "St. Kliment Ohridski"* (1), Proceedings of the International Conference "Constructive Theory of Functions" (2). These publications cover in general the results included in the thesis. A list of Candidate's results with 14 citations since 2010 is

attached and most of them appeared in journals with impact factor. The articles and citations mentioned in this procedure have not been used in previous procedures for the award of the PhD or for the academic position of Associate Professor. The results have been presented at a number of international conferences, as evidenced by the list of papers presented in Draganov's CV.

5. Qualities of the abstract

The abstract of the dissertation has been prepared very precisely, it is comprehensive and gives a very good overview of the overall dissertation. It contains brief historical notes on the subject, main definitions of the *K*-functionals and moduli of smoothness used in the dissertation, the most important results obtained, comments and the literature cited therein.

6. Critical notes and recommendations

I have no critical comments on the dissertation and the attached research papers.

7. Conclusion

Having become acquainted with the DSci Thesis presented in the procedure and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific and applied contributions contained therein, **I confirm** that the presented DSci Thesis and the scientific publications to it, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the Act on Development of the Academic Staff in the Republic of Bulgaria, the Rules for its Implementation and the corresponding Rules at the Sofia University "St. Kliment Ohridski" (FMI-SU) for acquisition by the Candidate of the scientific degree "Doctor of Science" in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics. In particular, the Candidate meets the minimal national requirements in the professional field and no plagiarism has been detected in the scientific papers submitted for the competition.

Based on the above, **I strongly recommend** the scientific jury to award Borislav Radkov Draganov the scientific degree „Doctor of Science" in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics, scientific specialty Mathematical Analysis.

Date: February 23, 2024   Signature: …………………………………..
                                               (Rumen Uluchev, Assoc, Prof, PhD)